Feedback control of the squeezing of the fluorescence light

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Abstract

Among the formulations of the theory of quantum measurements in continuous time, quantum trajectory theory is very suitable for the introduction of measurement based feedback and closed loop control of quantum systems. In this paper we present such a construction in the concrete case of a two-level atom stimulated by a coherent, monochromatic laser. In particular, we show how fast feedback à la Wiseman and Milburn can be introduced in the formulation of the theory. Then, the spectrum of the free fluorescence light is studied and typical quantum phenomena, squeezing and sub-natural line-narrowing, are presented.

1   Introduction

The theory of continuous measurements and filtering for quantum systems [6, 1] has opened the possibility of a consistent introduction of measurement based feedback in continuous time and, so, of closed loop control of quantum systems. The formulation of continuous measurement theory adopted in the paper is the one based on stochastic differential equations, commonly known in quantum optics as quantum trajectory theory [7].

Photo-detection theory in continuous time has been widely developed inside quantum trajectories [5, 8] and applied, in particular, to the fluorescence light emitted by a single trapped two-level atom stimulated by a coherent monochromatic laser. As well as various feedback schemes on the atom evolution, based on the outcoming photocurrent, have been proposed [9]. However the introduction and the analysis of feedback have been mainly focused on the control of the atom. The typical aim was to drive the atom to a preassigned asymptotic state.

Here, we are interested not only in the atom, but also, and mainly, in the emitted light. Of course, a detection of the emitted light can be seen as a direct observation of the emitted light as well as an indirect observation of the emitting atom mediated by the fluorescence light. Our aim is to employ control and feedback processes to enhance the squeezing properties of the fluorescence light. The squeezing of the fluorescence light can be checked by homodyne detection

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and spectral analysis of the output current. We consider the mathematical description of photo-detection based on quantum trajectories, as it is suitable both to consistently compute the homodyne spectrum of fluorescence light, and to introduce feedback and control in the mathematical formulation. We study how the squeezing depends on the various control parameters, how feedback mechanisms can be successfully introduced and used to enhance the squeezing [4, 3]. We consider only the Markovian feedback scheme à la Wiseman-Milburn [9], which is simple, but flexible enough to give a physically interesting model of closed loop control. We also show that, under certain conditions, feedback can produce an effect of line narrowing in the homodyne spectrum, similarly to what happens to an atom stimulated by squeezed light.

Apart from the new results on enhancing squeezing by feedback and on line narrowing, there is a conceptual interest in using quantum trajectory theory in deriving the homodyne spectrum. The traditional approach to homodyne or heterodyne spectrum is to define, by some analogy with the classical case, a suitable quantum correlation function for the outgoing electromagnetic field. Then, the spectrum is defined to be the Fourier transform of such a correlation function. Finally, the quantum fields are eliminated in favour of the atomic variables and the "quantum regression theorem" is used to get the final result. On the other side, quantum trajectory theory is based on quantum measurement theory and gives the output of the measurement in continuous time together with its distribution, consistently with principles of quantum mechanics. As the output is a stochastic process, its spectrum can be rigorously defined by probability theory by using its distribution. Moreover, thanks to the consistency of quantum continuous measurement theory, even when the quantum fields have been already eliminated (so that quantum regression theorem is implicitly already contained), the output can be seen not only as an observation of the system, but also as the result of an observation of the output field which mediates the measurement. To succeed in describing a quantum effect such as squeezing of the fluorescence light by using quantum trajectory theory is to show that such a theory, in spite of its "classical flavour", is fully quantum mechanical.

We consider a trapped two-level atom with Hilbert space $\mathbb{C}^2$. Denoting by $\vec{\sigma}$ the vector $(\sigma_x, \sigma_y, \sigma_z)$ of the Pauli matrices, let the free Hamiltonian of the atom be $\omega_0 \sigma_z / 2$, with resonance frequency $\omega_0 > 0$, and let the lowering and rising operators be σ_- and σ_+. Let the eigenprojectors of σ_z be denoted by P_+ and P_- and, for every angle ϕ, let us introduce the unitary selfadjoint operator

$$\sigma_\phi = \mathrm{e}^{\mathrm{i}\phi}\, \sigma_- + \mathrm{e}^{-\mathrm{i}\phi}\, \sigma_+ = \cos\phi\, \sigma_x + \sin\phi\, \sigma_y.$$

2 The dynamics

We start by giving the theory in the case of no feedback and then we show how to introduce it, according to the scheme of [9].

2.1 No feedback

We admit an open Markovian evolution for the atom, subjected to 'dephasing' effects and to interactions both with a thermal bath and with the electromagnetic field, via absorption and emission of photons. The atom is stimulated by a

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coherent monochromatic laser. We denote by $\gamma > 0$ the natural linewidth of the atom, by $\Omega \geq 0$ the Rabi frequency, by $\omega > 0$ the frequency of the stimulating laser and by $\Delta\omega = \omega_0 - \omega$ the detuning. Other parameters are the intensities of the dephasing and thermal effects, $k_{\mathrm{d}} \geq 0$ and $\overline{n} \geq 0$. The state η_t is governed by a Master equation, which turns out to be time-homogeneous in the rotating frame (which we adopt here and in the rest of the paper):

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\eta_t = \mathcal{L}\eta_t\,,$$

where

$$\mathcal{L}\rho = -\mathrm{i}\left[\frac{\Delta\omega}{2}\,\sigma_z + \frac{\Omega}{2}\,\sigma_x\,,\,\rho\right] + \gamma k_{\mathrm{d}}\left(\sigma_z\,\rho\,\sigma_z - \rho\right)$$
$$+ \gamma\overline{n}\left(\sigma_+\,\rho\,\sigma_- - \frac{1}{2}\left\{P_-\,,\,\rho\right\}\right) + \gamma(\overline{n}+1)\left(\sigma_-\,\rho\,\sigma_+ - \frac{1}{2}\left\{P_+\,,\,\rho\right\}\right).$$

Note that the intensity of the laser enters only in the effective Hamiltonian through the Rabi frequency Ω.

The measuring apparatus is made by two homodyne detectors. Part of the emitted light reaches the detectors and part is lost in the free space. The fraction of light detected by one of the detectors depends on its efficiency, on the spanned solid angle and can eventually be enhanced by using a focussing mirror. So, the fluorescence light is divided in three parts according to the direction of propagation: we call *side channel k* ($k = 1, 2$) the directions reaching the detector k, and *forward channel* those of the lost light. The stimulating laser is well collimated in such a way that it does not hit the detectors; so, we can say that it acts in the forward channel. We denote the effective fractions of light emitted in the forward and in the two side channels by $|\alpha_0|^2$, $|\alpha_1|^2$, $|\alpha_2|^2$, respectively; obviously, $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1$. For $k = 1, 2$, we can also interpret $|\alpha_k|^2$ as the total efficiency of the detector k. Moreover, the initial phase of the local oscillator in each detector is denoted by ϑ_k and it is included in the parameter $\alpha_k \in \mathbb{C}$ by setting $\vartheta_k = \arg \alpha_k$. To change ϑ_k means to change the measuring apparatus.

We can condition the evolution of the atom on the continuous monitoring of the photocurrents. The two homodyne photocurrents I_1 and I_2 and the conditional state of the atom ρ_t (a posteriori state) are stochastic processes whose distributions depend on the initial state of the atom and on the parameters introduced so far. In particular the atom has still a Markovian evolution, even if stochastic. Let us introduce the latter by means of the *linear* stochastic Master equation

$$\mathrm{d}\sigma_t = \mathcal{L}\sigma_t\,\mathrm{d}t + \sqrt{\gamma}\sum_{j=1}^{2}\mathcal{R}[\alpha_j\sigma_-]\sigma_t\,\mathrm{d}W_j(t), \tag{1}$$

where, for every matrix a, the superoperator $\mathcal{R}[a]$ is

$$\mathcal{R}[a]\rho = a\,\rho + \rho\,a^*,$$

and where W_1 and W_2 are two independent standard Wiener processes in some reference probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. In all these equations the initial condition is a statistical operator $\varrho_0 = \eta_0 = \sigma_0$ and the solution σ_t of (1) is a positive operator valued stochastic process.

Then, the homodyne photocurrents are the (generalised) stochastic processes $I_j(t) = \dot{W}_j(t)$, while the *a posteriori state* is

$$\rho_t := \frac{\sigma_t}{p_t}\,, \qquad \text{with} \quad p_t = \operatorname{Tr}\{\sigma_t\}.$$

The important point is that the physical distribution of the homodyne currents and the a posteriori states in the time interval $[0, T]$ is given by the new probability $\mathbb{P}_T(\mathrm{d}\omega) = p_T(\omega)\mathbb{Q}(\mathrm{d}\omega)$: the quantity $p_T = \operatorname{Tr}\{\sigma_T\}$ is the density of the *physical probability* with respect to the reference probability. By the fact that p_t, $t \geq 0$, turns out to be a $\mathbb{Q}$-martingale, we have that the physical probabilities do not depend on the final time T.

Of course, we could switch on the detectors, but decide to ignore the outputs. This should not modify the evolution of the atom and, indeed, for every $t \leq T$,

$$\eta_t = \int_\Omega \sigma_t(\omega)\,\mathbb{Q}(\mathrm{d}\omega) = \int_\Omega \rho_t(\omega)\,\mathbb{P}_T(\mathrm{d}\omega).$$

Thanks to the linearity of (1), we can introduce the stochastic evolution map (or propagator) $\mathcal{A}(t, s)$, satisfying

$$\mathcal{A}(t + \mathrm{d}t, t) - \operatorname{Id} = \mathcal{L}\mathrm{d}t + \sqrt{\gamma}\sum_{j=1}^{2}\mathcal{R}[\alpha_j\sigma_-]\mathrm{d}W_j(t),$$

so that $\sigma_t = \mathcal{A}(t, 0)\,\sigma_0$. Here, Id is the identity map on the space of 2×2 complex matrices.

2.2   Introduction of the feedback

We introduce a feedback scheme based on I_1, in order to modify the properties of the fluorescence light in channel 2. We check such properties by analysing the properties of I_2, but, of course, we could remove the second homodyne detector and employ the light emitted in channel 2 for other purposes. Let us call it *free* light. Our scheme is summarised by the following picture, where "h. det." means "homodyne detector":
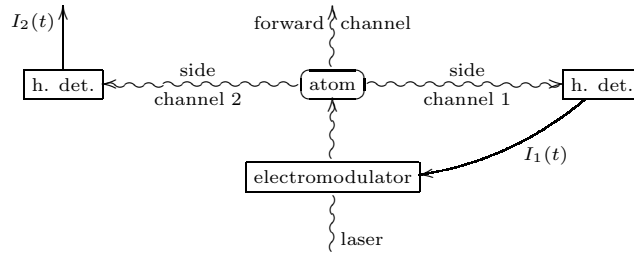
Assuming instantaneous feedback, the amplitude of the stimulating laser is modified by adding a term proportional to I_1, with the same frequency ω and with initial phase possibly different from that of the original laser. Let this phase difference be $\varphi \in [0, \pi)$. As the laser intensity appears only in the Hamiltonian part of the evolution (1), the effect of the feedback is to give rise to a new

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Hamiltonian term $I_1(t)\,M$ with $M = c\sqrt{\gamma}\,\sigma_\varphi$, where $c \in \mathbb{R}$ controls the intensity of the feedback. Let us deduce the modified evolution equation. By defining the map $\mathcal{M}\rho = -\mathrm{i}[M,\rho]$, the contribution of the feedback to the propagator in an infinitesimal interval turns out to be

$$\mathrm{e}^{\mathcal{M}\,\mathrm{d}W_1(t)} = \mathrm{Id} + \mathcal{M}\,\mathrm{d}W_1(t) + \frac{1}{2}\mathcal{M}^2\mathrm{d}t.$$

Taking into account that the feedback must act after the signal is produced, the new infinitesimal propagator is

$$\mathcal{A}_{\mathrm{f}}(t+\mathrm{d}t, t) = \mathrm{e}^{\mathcal{M}\,\mathrm{d}W_1(t)} \circ \mathcal{A}(t+\mathrm{d}t, t).$$

By the Ito rules we obtain

$$\mathcal{A}_{\mathrm{f}}(t+\mathrm{d}t, t) - \mathrm{Id} = \mathcal{L}_{\mathrm{f}}\mathrm{d}t$$
$$+ \sqrt{\gamma}\mathcal{R}[\alpha_1\,\sigma_- - \mathrm{i}c\,\sigma_\varphi]\,\mathrm{d}W_1(t) + \sqrt{\gamma}\mathcal{R}[\alpha_2\,\sigma_-]\,\mathrm{d}W_2(t),$$

where

$$\begin{aligned}
\mathcal{L}_{\mathrm{f}}\rho &= -\mathrm{i}\left[\frac{\Delta\omega_c}{2}\,\sigma_z + \frac{\Omega}{2}\,\sigma_x\,,\,\rho\right] + \gamma k_{\mathrm{d}}\,(\sigma_z\,\rho\,\sigma_z - \rho) \\
&+ \gamma\overline{n}\left(\sigma_+\,\rho\,\sigma_- - \frac{1}{2}\{P_-\,,\,\rho\}\right) \\
&+ \gamma(\overline{n}+1-|\alpha_1|^2)\left(\sigma_-\,\rho\,\sigma_+ - \frac{1}{2}\{P_+\,,\,\rho\}\right) \\
&+ \gamma(\alpha_1\,\sigma_- - \mathrm{i}c\,\sigma_\varphi)\,\rho\,(\overline{\alpha}_1\,\sigma_+ + \mathrm{i}c\,\sigma_\varphi) \\
&- \frac{\gamma}{2}\left\{\left(|\alpha_1|^2 - 2c|\alpha_1|\sin(\vartheta_1-\varphi)\right)P_+ + c^2\,,\,\rho\right\},
\end{aligned}$$

$$\Delta\omega_c = \Delta\omega + c\,\gamma\,|\alpha_1|\cos(\vartheta_1-\varphi) \in \mathbb{R}.$$

Thus, thanks to the features of the particular feedback introduced, we get another linear stochastic Master equation

$$\mathrm{d}\sigma_t = \mathcal{L}_{\mathrm{f}}\sigma_t\,\mathrm{d}t + \sqrt{\gamma}\mathcal{R}[\alpha_1\,\sigma_- - \mathrm{i}c\,\sigma_\varphi]\sigma_t\,\mathrm{d}W_1(t) + \sqrt{\gamma}\mathcal{R}[\alpha_2\,\sigma_-]\sigma_t\,\mathrm{d}W_2(t). \quad (2)$$

By Girsanov theorem, a fundamental result of stochastic calculus, it is possible to prove that under the physical probability the output homodyne currents can be written as

$$I_j(t)\mathrm{d}t = \mathrm{d}W_j(t) = \mathrm{d}B_j(t) + v_j(t)\mathrm{d}t,$$

where $B_1(t)$, $B_2(t)$ $(t \in [0,T])$ are two independent standard Wiener processes under $\mathbb{P}_T$ and

$$v_1(t) = 2\sqrt{\gamma}\,\mathrm{Re}\,\mathrm{Tr}\left\{(\alpha_1\sigma_- - \mathrm{i}c\sigma_\varphi)\,\rho_t\right\}, \qquad v_2(t) = 2\sqrt{\gamma}\,\mathrm{Re}\,(\alpha_2\,\mathrm{Tr}\,\{\sigma_-\rho_t\}).$$

Then the mean function of I_2 is

$$\mathbb{E}_{\mathbb{P}_T}[I_2(t)] = v_2(t).$$

Moreover, by using techniques based on characteristic functionals, explicit expressions for the higher moments of the output have been obtained [3]. In particular, the auto-correlation of I_2 is

$$\mathbb{E}_{\mathbb{P}_T}[I_2(t)\,I_2(s)] = \delta(t-s) + \gamma\,\operatorname{Tr}\Big\{\mathcal{R}[\alpha_2\sigma_-]$$
$$\circ\,\mathrm{e}^{|t-s|\mathcal{L}_f}\circ\mathcal{R}[\alpha_2\sigma_-]\circ\mathrm{e}^{\min\{s,t\}\mathcal{L}_f}\,\varrho_0\Big\}. \quad (3)$$

The expression above is similar to what is obtained in traditional approaches through the quantum regression theorem; the general theory of continuous measurements guarantees that it is a positive definite function of s and t as it must be for an auto-correlation function.

3   The spectrum of the free light in channel 2

Let us consider now the spectrum of the light in channel 2, the fluorescence light detected but not involved in the feedback loop. The homodyne current $I_2(t)$ is a stochastic process and it can be proved that it is asymptotically stationary. Then, the spectrum $S_2(\mu)$ is the Fourier transform of the autocorrelation function for long times. This classical definition can be recast in the following way:

$$S_2(\mu) = S_{\mathrm{el}}(\mu) + S_{\mathrm{inel}}(\mu),$$

$$S_{\mathrm{el}}(\mu) = \lim_{T\to+\infty}\frac{1}{T}\left|\mathbb{E}_{\mathbb{P}_T}\left[\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\,I_2(s)\,\mathrm{d}s\right]\right|^2,$$

$$S_{\mathrm{inel}}(\mu) = \lim_{T\to+\infty}\frac{1}{T}\Bigg\{\mathbb{E}_{\mathbb{P}_T}\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\,I_2(s)\,\mathrm{d}s\right|^2\right]$$
$$-\left|\mathbb{E}_{\mathbb{P}_T}\left[\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\,I_2(s)\,\mathrm{d}s\right]\right|^2\Bigg\} \geq 0, \quad (4)$$

where the spectrum is decomposed in the *elastic* and *inelastic* parts. By explicit computations on can see that $S_{\mathrm{el}}(\mu)$ is proportional to a Dirac delta centred in zero (as we are working in the rotating frame, $\mu = 0$ corresponds to resonance with the frequency of the stimulating laser). Instead, $S_{\mathrm{inel}}(\mu)$ is the limit of the normalised variance of the Fourier transform of the photocurrent I_2. An explicit expression can be obtained from the expression of the second moments (3). The asymptotic behaviour of the atomic a priori state $\eta_t = \mathrm{e}^{\mathcal{L}_f t}\varrho_0$ ensures that the limit defining the spectrum exists and that it is independent of the initial state ϱ_0 of the atom. We get

$$S_{\mathrm{inel}}(\mu) = 1 + 2\gamma|\alpha_2|^2\,\vec{s}\cdot\left(\frac{A}{A^2+\mu^2}\,\vec{t}\right), \quad (5)$$

where $\vec{s} = \big(\cos\vartheta_2,\quad \sin\vartheta_2,\quad 0\big)$, the matrix A has matrix elements

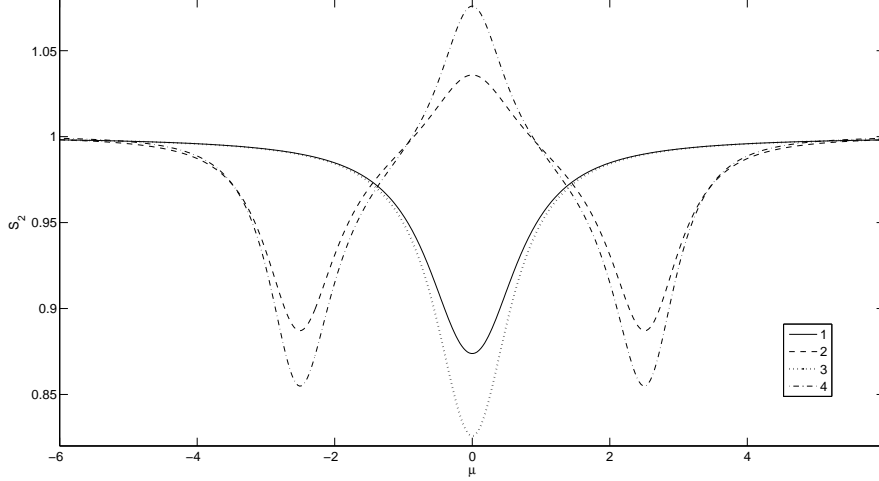$$a_{13} = a_{31} = 0, \qquad a_{23} = -a_{32} = \Omega,$$

Figure 1: $S_{\text{inel}}(\mu)$ with $\gamma = 1$ and: (1) $\Delta\omega = 0$, $c = 0$, $\Omega = 0.2976$, $\vartheta_2 = -\pi/2$; (2) $\Delta\omega = 1.8195$, $c = 0$, $\Omega = 1.7988$, $\vartheta_2 = -0.1438$; (3) $\Delta\omega = 0$, $c = 0.0896$, $\Omega = 0.2698$, $\vartheta_1 = \pi/2$, $\vartheta_2 = -\pi/2$, $\varphi = 0$; (4) $\Delta\omega = 1.6920$, $c = 0.1326$, $\Omega = 1.9276$, $\vartheta_1 = 2.8168$, $\vartheta_2 = -0.0851$, $\varphi = 1.2460$.

$$a_{11} = \gamma\Big(\frac{1}{2} + \overline{n} + 2k_{\mathrm{d}} + 2c|\alpha_1|\cos\vartheta_1\sin\varphi + 2c^2\sin^2\varphi\Big),$$

$$a_{12} = \Delta\omega_c - \gamma\Big(c|\alpha_1|\cos(\vartheta_1 + \varphi) + c^2\sin 2\varphi\Big),$$

$$a_{21} = -\Delta\omega_c - \gamma\Big(c|\alpha_1|\cos(\vartheta_1 + \varphi) + c^2\sin 2\varphi\Big),$$

$$a_{22} = \gamma\Big(\frac{1}{2} + \overline{n} + 2k_{\mathrm{d}} - 2c|\alpha_1|\sin\vartheta_1\cos\varphi + 2c^2\cos^2\varphi\Big),$$

$$a_{33} = \gamma\Big(1 + 2\overline{n} - 2c|\alpha_1|\sin(\vartheta_1 - \varphi) + 2c^2\Big),$$

$$\vec{t} = \mathrm{Tr}\Big\{\big(\mathrm{e}^{\mathrm{i}\vartheta_2}\,\sigma_-\,\eta_{\mathrm{eq}} + \mathrm{e}^{-\mathrm{i}\vartheta_k}\,\eta_{\mathrm{eq}}\,\sigma_+ - \mathrm{Tr}\{\sigma_{\vartheta_2}\,\eta_{\mathrm{eq}}\}\,\eta_{\mathrm{eq}}\big)\,\vec{\sigma}\Big\}.$$

and $\eta_{\mathrm{eq}} = \lim\limits_{t\to+\infty}\mathrm{e}^{\mathcal{L}_f t}\varrho_0$.

In the case $|\alpha_2| = 0$ we get $S_{\text{inel}}(\mu) = 1$, which is the spectrum of a pure white noise. Indeed in this case no fluorescence light reaches the detector and we see only the spectrum of the fluctuations of the local oscillator, interpreted as shot noise.

3.1   Squeezing

In general it is possible to give a description of the output I_2 in terms of a measurement of some Bose quantum fields, to show that the value of $S_{\text{inel}}(\mu; \vartheta_2)$ is the variance of a quadrature of the field in channel 2, the value of $S_{\text{inel}}(\mu; \vartheta_2 + \pi/2)$ is the variance of the conjugate quadrature [2]. Then, the *Heisenberg uncertainty relations* imply that

$$S_{\text{inel}}(\mu; \vartheta_2)\, S_{\text{inel}}(\mu; \vartheta_2 + \pi/2) \geq 1.$$

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If $S_{\mathrm{inel}}(\mu; \vartheta_2) < 1$ for some μ and θ_2, the field is said to be *squeezed*.

Some typical spectra, showing well pronounced squeezing and produced by suitable choices of the control parameters, are given in Figure 1. The position of the minimum can be *tuned* by a suitable choice of the control parameters; in Figure 1 the minima are in $\mu = 0$ and in $\mu = \pm 2.5$, with and without feedback. The lines (1) and (2) are without feedback and the lines (3) and (4) are with feedback. Here and in all the other graphical examples we take $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$, $\overline{n} = 0$ and $k_{\mathrm{d}} = 0$.

Note that the Heisenberg uncertainty relations give rise to peaks in the complementary field quadratures, as shown in Figure 2.

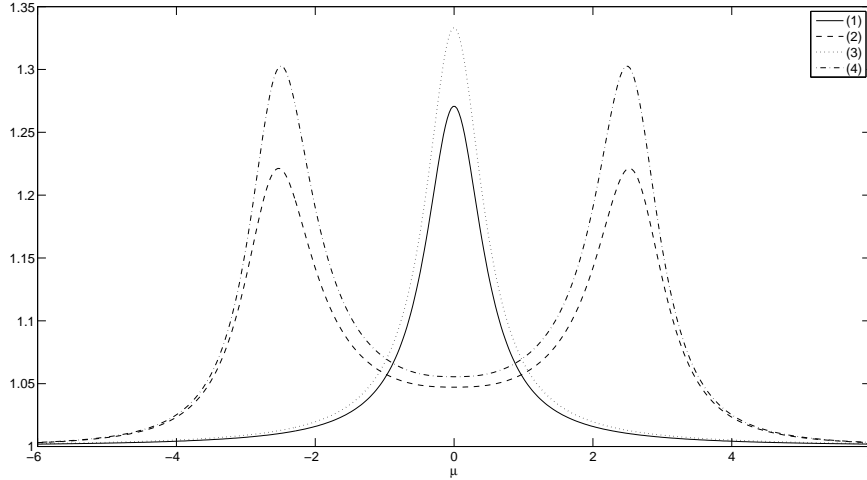Figure 2: $S_{\mathrm{inel}}(\mu)$ with all the parameters as in the previous figure, but $\vartheta_2 \rightarrow \vartheta_2 + \pi/2$.

3.2 Line-narrowing

By feedback control another quantum effect can be produced, the line-narrowing. After the first observation of squeezing, Gardiner predicted that stimulating a two-level atom with squeezed light would inhibit the phase decay of the atom. The squeezed light would break the equality between the transverse decay rates for the two quadratures of the atom and one decay rate could be made arbitrarily small, producing an observable narrow line in the spectrum of the atomic fluorescence light. This was seen as a "direct effect of squeezing" and thus as a measure of the squeezing of the incident light. Nevertheless, Wiseman showed that this atomic line-narrowing is not only characteristic of squeezed light, but it can also be produced by immersing a two-level atom in 'in-loop squeezed' light. The difference is that in the Gardiner case the other decay rate becomes larger, while in the Wiseman case it is left unchanged.

Now we can show that the same atomic line-narrowing can be obtained stimulating a two-level atom even with non-squeezed light, that is with a coherent monochromatic laser in presence of a (Wiseman-Milburn) feedback scheme based on the (homodyne) detection of the fluorescence light. This effect is ob-
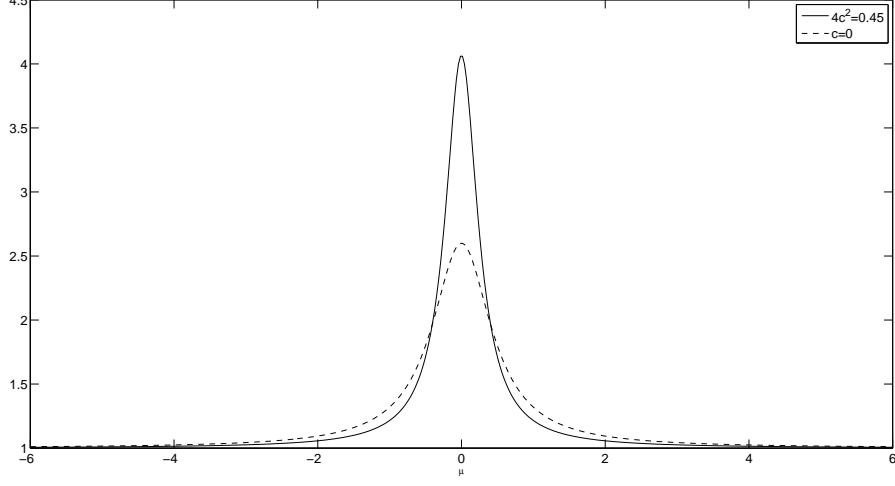
Figure 3: $S_{\text{inel}}(\mu)$ for $c = |\alpha_1|/2$ or $c = 0$ and $\Omega = 2$, $p_1 = p_2 = 0.45$, $\Delta\omega = 0$, $\varphi = \frac{\pi}{2}$, $\vartheta_1 = \pi$, $\vartheta_2 = 0$, $\gamma = 1$.

tained within the model described up to now, in a different region of the control parameters.

Let us take

$$\varphi = \frac{\pi}{2}, \qquad \vartheta_1 = \pi, \qquad \vartheta_2 = 0, \qquad \Delta\omega = 0, \qquad \overline{n} = 0, \qquad k_{\text{d}} = 0.$$

Then, in the case of no feedback $c = 0$ one obtains a spectrum which is a Lorentzian peak with width γ:

$$S_{\text{inel}}(\mu) = 1 + \frac{2 |\alpha_2|^2 \Omega^2}{\gamma^2 + 2\Omega^2} \frac{\gamma^2}{\mu^2 + \gamma^2/4}.$$

Instead, with the optimal choice of the feedback, $c = |\alpha_1|/2$, we get a Lorentzian of width $\left(1 - |\alpha_1|^2\right)\gamma$:

$$S_{\text{inel}}(\mu) = 1 + \frac{|\alpha_2|^2 \left(\frac{|\alpha_1|^2}{2}\gamma^2 + 2\Omega^2\right)}{\left(1 - \frac{|\alpha_1|^2}{2}\right)\gamma^2 + 2\Omega^2} \frac{\left(1 - |\alpha_1|^2\right)\gamma^2}{\mu^2 + \left(1 - |\alpha_1|^2\right)^2 \gamma^2/4}$$

This is the sub-natural line-narrowing effect, shown in Figure 3.

4   Conclusions

The new results presented either are of general conceptual relevance, either clarify the behaviour of fluorescence light and atom under coherent driving and feedback.

First of all we have shown how to introduce the spectrum in quantum trajectory theory through its classical definition (4), without ad hoc quantum definitions, but in agreement with the axiomatic structure of quantum measurement

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theory and with probability theory. We have verified that this agreement with quantum measurement theory ensures consistency with the existence of the traced out quantum field, so that Heisenberg uncertainty relations hold for such spectra and the squeezing of the output field can be analysed.

We have shown how feedback (at least in the form à la Wiseman-Milburn) modifies the spectrum of the free fluorescence light and can enhance its squeezing. Many physical effects are introduced all together: detuning, thermal and dephasing effects, not perfect detection efficiency, control. The final formula for the spectrum (5) is given with all the parameters introduced by these effects.

Finally, we have shown that feedback can produce line-narrowing in the free fluorescence light, even if the atom is not illuminated by squeezed light, but only by coherent light.

References

[1] A. Barchielli, V. P. Belavkin, *Measurements continuous in time and a posteriori states in quantum mechanics*, J. Phys. A: Math. Gen. **24** (1991) 1495–1514; arXiv:quant-ph/0512189.

[2] A. Barchielli, M. Gregoratti, *Quantum continual measurements: the spectrum of the output.* In J. C. García, R. Quezada, S. B. Sontz (eds.), *Quantum Probability and Related Topics*, Quantum Probability Series QP-PQ Vol. 23 (World Scientific, Singapore, 2008) pp. 63–76; arXiv:0802.1877v1 [quant-ph].

[3] A. Barchielli, M. Gregoratti, *Quantum Trajectories and Measurements in Continuous Time: The Diffusive Case*, Lect. Notes Phys. **782** (Springer, Berlin, 2009); DOI 10.1007/978-3-642-01298-3.

[4] A. Barchielli, M. Gregoratti, M. Licciardo, *Feedback control of the fluorescence light squeezing*, Europhysics Letters (EPL) **85** (2009) 14006; doi: 10.1209/0295-5075/85/14006; arXiv:0804.0085v1 [quant-ph].

[5] A. Barchielli, A. M. Paganoni, *Detection theory in quantum optics: Stochastic representation*, Quantum Semiclass. Opt. **8** (1996) 133–156.

[6] V. P. Belavkin, *Nondemolition measurements, nonlinear filtering and dynamic programming of quantum stochastic processes.* In A. Blaquière (ed.), *Modelling and Control of Systems*, Lecture Notes in Control and Information Sciences **121** (Springer, Berlin, 1988) pp. 245–265.

[7] H. J. Carmichael, *Statistical Methods in Quantum Optics 2. Non-Classical Fields* (Springer, Berlin, 2008).

[8] P. Zoller, C. W. Gardiner, *Quantum noise in quantum optics: the stochastic Schrödinger equation.* In S. Reynaud, E. Giacobino, J. Zinn-Justin (eds.), *Fluctuations quantiques, (Les Houches 1995)* (North-Holland, Amsterdam 1997) pp. 79-136.

[9] H. M. Wiseman, G. J. Milburn, *Quantum theory of optical feedback via homodyne detection*, Phys. Rev. Lett. **70** (1993) 548–551.